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Semi-rigid Classes of Cotorsion-Free Abelian Groups

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1. INTRODUCTION

A class $\mathscr{H}$ of abelian groups is rigid if $\operatorname{Hom}(A, B) = 0$ for pairs $A, B \in \mathscr{H}$ of different groups A, B. We will say that $\mathscr{H}$ is *semi-rigid* if $\operatorname{Hom}(A, B) \neq 0$ implies $\operatorname{Hom}(B, A) = 0$ for all pairs $A, B \in \mathscr{H}$ of different groups A, B. In order to answer some open problems we are interested in such systems $\mathscr{H}$ which are not sets. We will simply say, that $\mathscr{H}$ is a *(semi-)rigid class* if $\mathscr{H}$ is not a set.

From a well-known result in model-theory we derive that the existence of a rigid class seems to be not provable in ZFC, e.g., Kanomori and Magidor [21, Sect. 17, pp. 196–203]. Hence one cannot expect to prove the existence of a rigid system which is a proper class. On the other hand, it is not hard to show that there are rigid classes of strongly-$|G|$-free groups G in ZFC + $V = L$; cf. also Section 5. A construction of a rigid class in $V = L$ is given in Dugas and Herden [7, Theorem 2.1]. Using methods from [4, 28] we will weaken their set theoretic hypothesis in Section 5. Large rigid systems are constructed in [1, 11, 25], using only ZFC.

However, in this paper we will show, using ZFC only, that there are always semi-rigid classes of abelian groups. More precisely we will establish

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a semi-rigid class $\mathscr{K}_G$ of cotorsion-free groups containing a prescribed cotorsion-free group $G \in \mathscr{K}_G$. The groups in $\mathscr{K}_G \backslash G$ can be choosen to be homogeneous of any idempotent type $\tau \neq (\infty, \infty, \ldots, \infty)$.

A group G is homogeneous of type τ if all its elements different from 0 have type τ. A group G is called cotorsion-free if 0 is the only cotorsion subgroup of G; cf. Göbel [13, p. 41]. Cotorsion-free groups are characterized by the following

THEOREM 1. ([15, 4]). *For a group G the following are equivalent*

(1) *G is cotorsion-free,*

(2) *G is torsion-free and $\mathbb{Q}$ and I_p are not direct summands of G for all primes p.*

(3) *G contains no subgroups isomorphic to $\mathbb{Q}, I_p, Z_p$ for all primes p.*

Here $\mathbb{Q}$ denotes the additive group of the quotient field of $\mathbb{Z}$ and I_p, Z_p are the additive groups of the rings of p-adic integers and of $\mathbb{Z}/p\mathbb{Z}$, respectively. The existence of a semi-rigid system $\mathscr{K}_G$ follows immediately from our main result which is the

THEOREM 2. *If μ is an arbitrary strong limit singular cardinal with $\mathrm{cf}(\mu) > \aleph_0$ and $\tau \neq (\infty, \ldots, \infty)$ is an idempotent type, there is a homogeneous cotorsion-free group G of cardinality μ and type τ with the following property.*

If $U \subseteq G$ with $|U| = \mu$ and G/U is cotorsion-free, then $U = G$.

Remarks concerning Theorem 2. (1) The result bears some similarity with the Jonsson-groups, constructed in Shelah [27]. However, the proof is completely different.

(2) If $\bar{\mathbb{Z}} = \prod_p I_p$, then an abelian group G will always have homomorphisms into $\mathbb{Z}, \mathbb{Q}$ or Z_p. This illustrates that it is also necessary to exclude cotorsion groups as subgroups of G/U in Theorem 2. It is not sufficient to assume that G/U is torsion-free and reduced.

(3) If we assume $V = L$ and also $|G/U| < |G|$, the theorem is due to Dugas and Herden [7]. In fact, Theorem 2 answers a question in [7, 8]. The construction in [7] follows Dugas and Göbel [4] and Shelah [28] using a modified step-lemma. In this case $|G|$ can be any regular cardinal $\neq \aleph_0$. We will come back to this case in Section 5.

(4) Using some refined set theoretic argument, the strong limit cardinal μ in Theorem 2 can be replaced by any cardinal μ such that $\mu^{\aleph_0} = \mu$ and $\mathrm{cf}(\mu) > \aleph_0$. The set theoretic concept for this will be provided in Shelah [29].

(5) Using arguments derived in Dugas and Göbel [6] it is easy to see

that Theorem 2 holds in the category of R-modules for Dedekind domains R which are neither fields nor complete discrete valuation rings.

Theorem 2 is used to derive the following results: The class of cotorsion-free groups is not cogenerated by a set of groups. Hence there is no set of groups such that all cotorsion groups can be obtained by iterated constructions of products, subgroups and extension starting with this set. This answers a question in Göbel and Wald [16, p. 221]. Results in torsion theories of abelian groups are derived which say that torsion classes generated by a single group and torsion classes cogenerated by a single group do not have small cardinality; compare Section 4. This answers a problem raised by Golan [17].

2. Some Set Theoretic Notations and Results

The set theoretic results have already been used for constructions of p-groups in Shelah [26] and in Dugas and Göbel [5] and for constructions of torsion-free groups in Dugas and Göbel [6]. A refinement of these combinatorial ideas will be developed in Shelah [29]. These can also be used, as in Section 3 for constructing groups of cardinalities that are not singular strong limits. However, we would like to mention that singular strong limit cardinals form a class and so we will obtain a class and not a set of the required groups.

For ease of reference we recall the set theoretic background of [5]. If $\{X_i, i \in I\}$ is a family of disjoint sets, then $\prod_{i \in I} X_i$ denotes the cartesian product. A cardinal κ is a strong limit cardinal if $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. From a theorem of Tarski we derive for strong limits κ that $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$; compare Jech [19, p. 50(6.21)]. Let $\mathrm{cf}(\kappa) = \inf\{|X|, X \subseteq \kappa$ and $\sup X = \kappa\}$ denote the cofinality of κ. Therefore we derive the

LEMMA 2.1. *If κ is a strong limit cardinal and $\kappa = \bigcup_{\nu < \mathrm{cf}(\kappa)} \kappa_\nu$, then* $|\prod_{\nu < \mathrm{cf}(\kappa)} \kappa_\nu| = 2^\kappa$.

Recall from Dugas and Göbel [5, Lemma 2.6] the

LEMMA 2.2. *Let κ be a cardinal and $\mathscr{F}, \mathscr{E}$ two sets of countable subsets of κ with the following properties*

(a) *The elements of $\mathscr{F}$ are almost disjoint, i.e., $a, b \in \mathscr{F}$ and $|a \cap b| = \aleph_0$ implies $a = b$.*

(b) $|\mathscr{E}| < |\mathscr{F}|$.

(c) $2^{\aleph_0} < |\mathscr{F}|$.

Then there is a subset $\mathscr{F}^*$ *of* $\mathscr{F}$ *such that* $|\mathscr{F}^*| = |\mathscr{F}|$ *and*

(d) $f \in \mathscr{F}^*$, $t \in \mathscr{E}$ *implies* $|f \cap t| < \aleph_0$.

We will also use the

"CIA-"Lemma 2.3. *Let* $X = \prod_{\alpha<\kappa} X_\alpha$ *be the cartesian product of sets* X_α *such that* $|X_\alpha|^{|\alpha|} = |X_\alpha| \leqslant |X_\beta|$ *if* $\alpha \leqslant \beta < \kappa$. *There is a subset* $F \subseteq X$ *such that*

(a) $|F| = |X|$.

(b) *If* $f, g \in F$ *and* $\sup\{\nu < \kappa, f(\nu) = \rho(\nu)\} = \kappa$ *then* $f = g$.

Proof. Cf. Dugas and Göbel [5, Lemma 2.7].

Lemma 2.4. *Let* λ *be a regular cardinal such that* $\rho^{\aleph_0} < \lambda$ *for all* $\rho < \lambda$. *If* Σ *is any system of countable subsets of* λ *of cardinality* λ, *then there is a subsystem* Σ' *of* Σ *with the same cardinality and a subset* $F \subseteq \lambda$ *such that* $B \cap C = F$ *for all different* $B, C \in \Sigma$.

This is a special case of the Δ-Lemma of Erdös and Rado [10, p. 468, Lemma 1], cf. also [5, Corollary 2.9].

Lemma 2.5. *Let* ρ *be a cardinal of cofinality* $>\aleph_0$ *and* F *a set of countable subsets of* ρ *such that* $|F| = \lambda^+$ *for some cardinal* $\lambda \geqslant \mathrm{cf}(\rho)$. *Then there is a subset* $F' \subseteq F$ *and a cardinal* $\rho' < \rho$ *such that* $\sup(f) \leqslant \rho'$ *for all* $f \in F'$ *and* $|F| = |F'|$.

Proof. Cf. [5, Lemma 2.5].

3. Construction of Homogeneous Cotorsion-Free Groups Which Have Only Small Subgroups with Cotorsion-Free Quotient

We will use standard notation as in Fuchs [12]. If G is torsion-free and S is a rank-1 pure subgroup, there are very many groups H (not cotorsion-free), containing a subgroup P which is pure-injective and having a non-zero homomorphism $\sigma: S \to P \subseteq H$. Therefore $\mathrm{Hom}(G, H) \neq 0$ in all these cases. This illustrates in a very strong sense the necessity of our assumption "cotorsion-free" for G/U in the next

Theorem 3.1. *If* μ *is a strong limit singular cardinal with* $\mathrm{cf}(\mu) > \aleph_0$ *there is a cotorsion-free homogeneous group* G *of rank* μ *such that* $U = G$ *for all subgroups* $U \subseteq G$ *with* $|U| = \mu$ *and* G/U *cotorsion-free.*

The result will follow from the next lemmata and propositions, using the following notation.

If $\alpha < \mu$, use α also as a generator for an infinite cyclic group. Hence $B = \bigoplus_{\alpha<\mu} \alpha\mathbb{Z}$ is a free abelian group. If $X \subseteq \mu$, let $B_X = \bigoplus_{\alpha\in X} \alpha\mathbb{Z}$; and B_X is contained in B. In particular $B_\lambda \subseteq B$ for $\lambda < \mu$ since ordinals $\lambda < \mu$ are subsets of μ. Then $\hat{B}_X$ will be the p-adic completion of B_X and in particular $\hat{B} = \hat{B}_\mu$. The group G will be constructed so that $B \subset G \subset \hat{B}$. This will be done by constructing generators ${}^\pi a_\alpha^\lambda \in \hat{B}$ by a transfinite induction on a well-ordered set $\mathscr{L}$ such that $G = \langle B, {}^\pi a_\alpha^\lambda / (\lambda, \alpha) \in \mathscr{L}, \pi \in I_p \rangle_*$. In this case the group G will be of homogeneous type $\tau = (\infty, \infty, \ldots, \infty, 0, \infty, \ldots, \infty)$, with an 0 at the place p. If we change the completions using also other primes, we easily see that we can also produce all types mentioned in Section 1. In order to be definite, we will concentrate on the p-adic completion. As usual $U_* \subset B$ means the smallest pure subgroup of B containing U. Since B is torsion-free, we know that $u_* \in U_*$ if and only if $u_* \in B$ and there is a positive integer n such that $nu_* \in U$. If $g \in \hat{B}$ we also write $g = \sum_{\alpha<\mu} \alpha g_\alpha$. Obviously $g_\alpha = 0$ for all $\alpha < \mu$ with at most countably many exceptions. If $n \in \omega$, then $g_\alpha \in p^n I_p$ for almost all $\alpha < \mu$. Elements in $\hat{B}$ are characterized by these two conditions. Recall also that $\hat{\mathbb{Z}} = I_p$ is the group of p-adic integers. If $g \in \hat{B}$, we denote the support of g by $[g]$. Thus $[g] = \{\alpha < \mu, g_\alpha \neq 0\}$. Let $\| g \|$ denote the supremum of g, i.e., $\| g \| = \sup[g]$. If $\lambda \in \mu$ and $g \in \hat{B}$, the element g is λ-high if and only if $[g] \subseteq \lambda$ and $\| g \| = \lambda$. Obviously, this is only possible if $\mathrm{cf}(\lambda) = \omega$. Hence let $\mathring{\mu} = \{\lambda < \mu, \mathrm{cf}(\lambda) = \omega, \mathrm{cf}(\mu) < \lambda, \lambda \text{ a strong limit cardinal}\}$. Recall that $\mathring{\mu}$ is a stationary subset of μ.

We will also use the following definition, which is important for our constructions.

Definition 3.2. Let $\lambda \in \mathring{\mu}$. A family $\{A_k, k \in \omega\}$ of disjoint subsets of B will be called λ-big if it has the following two properties

(i) There exists a strictly increasing sequence of cardinals $\lambda_k \in \mathring{\mu}$ such that $\lim_{k\in\omega} \lambda_k = \lambda$, $|A_k| \geqslant (\lambda_k^{\aleph_0})^+$ and $A_k \subseteq \hat{B}_{\lambda_{k+1} \setminus \lambda_k}$ for all $k \in \omega$.

(ii) The supports of elements of $A = \bigcup_{k\in\omega} A_k$ are pairwise disjoint, i.e., if $a, b \in A$ and $[a] \cap [b] \neq \varnothing$ then $a = b$.

Our next lemma shows, that λ-big families are really big. The lemma follows immediately from (2.1).

Lemma 3.3. *If $\lambda \in \mathring{\mu}$ and $\{A_k, k \in \omega\}$ is a λ-big family, then* $|\prod_{k\in\omega} A_k| = 2^\lambda$.

Next we will consider for a fixed $\lambda \in \mathring{\mu}$ all pairs $(b, A_k, k \in \omega)$ of elements $b \in \hat{B}$ with $\|b\| < \lambda$ and λ-big families $\{A_k, k \in \omega\}$. An easy cardinality argument shows that there are 2^λ such pairs. Hence we can label all such pairs as $(b_\alpha^\lambda, {}^kA_\alpha^\lambda, k \in \omega)$ with $\alpha < 2^\lambda$ and we will use $\mathscr{L} = \{(\lambda, \alpha), \lambda \in \mathring{\mu}, \alpha < 2^\lambda\}$ as an indexing set. The set $\mathscr{L}$ will be ordered by the lexicographical well-ordering on the pairs $(\lambda, \alpha) \in \mathscr{L}$. Therefore we can apply a transfinite induction on $\mathscr{L}$ to obtain the required generators of G and for the definition of a subset $C \subseteq \mathscr{L}$. We assume that such generators ${}^\pi a_\beta^\kappa$ are already constructed and that we know already whether $(\kappa, \beta) \in C$ or $(\kappa, \beta) \in \mathscr{L} \setminus C$ for all $(\kappa, \beta) < (\lambda, \alpha)$.

Let $G_\alpha^\lambda = \langle B, {}^\pi a_\beta^\kappa / (\kappa, \beta) \in \mathscr{L}, \ (\kappa, \beta) < (\lambda, \alpha), \ \pi \in I_p \rangle_*$. We say that $(\lambda, \alpha) \in C$ if and only if $\bigcup_{k \in \omega} {}^kA_\alpha^\lambda \subseteq G_\alpha^\lambda$, and $b_\alpha^\lambda \in G_\alpha^\lambda$. An element $a_\alpha^\lambda \in \hat{B}_\lambda$ will be called rigid at stage (λ, α) if a_α^λ is λ-high and

$$\|[{}^\pi a_\beta^\kappa] \cap [a_\alpha^\lambda]\| < \lambda \qquad \text{for all} \quad (\kappa, \beta) < (\lambda, \alpha), \pi \in I_p. \tag{$\dagger$}$$

If $(\lambda, \alpha) \in \mathscr{L} \setminus C$, choose any a_α^λ which is rigid at stage (λ, α) and put ${}^\pi a_\alpha^\lambda = a_\alpha^\lambda$ for all $\pi \in I_p$. If $(\lambda, \alpha) \in C$, let $b = b_\alpha^\lambda$ and $A_k = {}^kA_\alpha^\lambda$ be the given elements. In this case choose

$$\left.\begin{array}{l}\text{a family } {}^\pi a_\alpha^\lambda = a_\pi + b \cdot \pi \text{ indexed by } \pi \in I_p \text{ of rigid} \\ \text{elements at stage } (\lambda, \alpha) \text{ such that } a_\pi \in \prod_{k \in \omega} A_k p^k \text{ and also} \\ \|[{}^\pi a_\alpha^\lambda] \cap [{}^\rho a_\alpha^\lambda]\| < \lambda \text{ for all } \pi \neq \rho \in I_p.\end{array}\right\} \tag{$\dagger\dagger$}$$

The next lemma will show that a family $\{{}^\pi a_\alpha^\lambda / \pi \in I_p\}$ always exists. This concludes the construction of $G = \langle B, \ {}^\pi a_\alpha^\lambda, \ (\lambda, \alpha) \in \mathscr{L}, \pi \in I_p \rangle_* = \bigcup_{(\lambda, \alpha) \in \mathscr{L}} G_\alpha^\lambda$. We will reserve the letter G for this group.

LEMMA 3.4. *Using the notation above we assume that the elements ${}^\pi a_\beta^\kappa$ for $(\kappa, \beta) < (\lambda, \alpha) \in \mathscr{L}$ are defined.* Then there exists a rigid family ${}^\pi a_\alpha^\lambda$ *for $\pi \in I_p$ satisfying* $(\dagger\dagger)$.

Proof. If $(\lambda, \alpha) \notin C$, let $A_k = B_{\lambda_{k+1} \setminus \lambda_k}$ for some sequence $\lambda_k < \lambda_{k+1} \in \mathring{\mu}$ with $\lim_{k \in \omega} \lambda_k = \lambda$. Since $|A_k| = \lambda_{k+1}$ and $\{A_k, k \in \omega\}$ is λ-big by (3.2), we will proceed as in the next case where $(\lambda, \alpha) \in C$. In this case we know that $\bigcup_{k \in \omega} {}^kA_\alpha^\lambda \subseteq G_\alpha^\lambda$ and $\{A_k = {}^kA_\alpha^\lambda, k \in \omega\}$ is λ-big. We derive from (3.3) that $|\prod_{k \in \omega} A_k \cdot p^k| = 2^\lambda$. Since $\lim_{k \in \omega} |A_k| = \lambda$, there are 2^λ elements in $\prod_{k \in \omega} A_k \cdot p^k$ which even multiplied with a p-adic number ($\neq 0$) are still rigid at stage (λ, α). Here we used Lemmata 2.2 and 2.3! Let $\mathscr{F}$ be this familly. Because $|\mathscr{F}| = 2^\lambda > 2^{\aleph_0} = |I_p|$, we can choose a subfamily of $\mathscr{F}$, which we label as $\{a_\pi / \pi \in I_p\}$ with $\|[a_\pi] \cap [a_\rho]\| < \lambda$ for all $\pi \neq \rho \in I_p$. Since $\|b\| < \lambda$, $\{a_\pi + b\pi / \pi \in I_p\}$ is also a family of rigid elements at stage (λ, α) and $(\dagger\dagger)$ is fulfilled. Hence ${}^\pi a_\alpha^\lambda = a_\pi + b\pi$ are the required elements in this case.

The following observation is a trivial consequence of the construction of our system of rigid elements ${}^\pi a_\alpha^\lambda$. The very elementary proof by induction is also given explicitly in Dugas and Göbel [6, (4.3) (1)]. Recall that B is a free group by definition and that the new elements ${}^{\pi}a_{\alpha}^{\lambda}$ are linearly independent modulo B.

OBSERVATION 3.5. (a) The set $B \cup \{{}^{\pi}a_{\alpha}^{\lambda}/(\lambda, \alpha) \in \mathscr{L}, \pi \in I_p\}$ generates a free abelian subgroup of G. In fact G is the purification of this group.

(b) If $v \in G$ and $\|v\| < \lambda$, then $v \in \bigcup_{\alpha < 2^{\lambda}} G_{\alpha}^{\lambda}$.

Next we want to show the following

LEMMA 3.6. *Let G be the constructed group and $U \subseteq G$ such that $|U| \geqslant \lambda \in \mathring{\mu}$. Then there is a strictly increasing sequence of cardinals $\{\lambda_n\}$ in $\mathring{\mu}$* and elements $\{y_i^n, i < (\lambda_n^{\aleph_0})^+\}$ *such that*

(0) $\lim_{n \in \omega} \lambda_n = \lambda$.

(1) $(\lambda_n^{\aleph_0})^+ < \lambda$.

(2) $y_i^n \in U$ *for all* $i < (\lambda_n^{\aleph_0})^+$ *and* $n \in \omega$.

(3) $[y_i^n] \subseteq \lambda_{n+1} \backslash \lambda_n$.

(4) $[y_i^n]$ *are pairwise disjoint for all* $i < ({}_n^{\aleph_0})^+$.

Using Definition 3.2 and notation $A_n = \{y_i^n/i < (\lambda_n^{\aleph_0})^+\}$ we derive from (3.6) the immediate

COROLLARY 3.7. *If $U \subseteq G$ and $|U| \geqslant \lambda \in \mathring{\mu}$, the group U contains a λ-big family.*

Proof of 3.6. The proof is similar to Shelah [26] and Dugas and Göbel [5]. In the first step we show

(a) There is a subset $U' \subseteq U$ such that $|U| = |U'|$ and the supports of elements in U' are pairwise disjoint.

From the Δ-Lemma 2.4 we obtain a subset $U_1 \subseteq U$ and a subset $F \subseteq \mu$, where $|F| \leqslant \aleph_0$ such that $|U_1| = |U|$ and $[u] \cap [v] = F$ for all $u \neq v \in U_1$. Remember that we build at most $2^{\aleph_0}$ different functions on the countable set F, with values in I_p. Since $|U_1| = |U| \geqslant \lambda > 2^{\aleph_0}$ there is a subset $U_2 \subseteq U_1$ such that $|U_2| = |U_1|$ and $u|_F = \text{const.}$ for all $u \in U_2$. Here $u|_F$ means the restriction of u for the subset F. Choose any partition $U_2 = U_3 \mathbin{\dot\cup} U_4$ of U_2 into two subsets of the same cardinality and let $': U_3 \to U_4$ be a bijection. Therefore $u - u'|_F = 0$ for all $u \in U_3$ and $U' = \{u - u'/u \in U_3\}$ is a set of elements with pairwise disjoint supports and $|U'| = |U_3| = |U|$.

In the final step we construct the required elements in (3.6) using induction on $n \in \omega$. Let $\{\mu_n \in \mathring{\mu}, n \in \omega\}$ any strictly increasing sequence with $\lim_{n \in \omega} \mu_n = \lambda$. Assume $\{y_i^m, i < (\lambda_m^{\aleph_0})^+\} = A_m$ for $m \leqslant n$ and also λ_m with $\mu_m < \lambda_m < \lambda$ for $m \leqslant n+1$ satisfying (1) to (4) have been constructed.

Let $T = \{u \in U', \ [u] \cap \lambda_{n+1} = \emptyset\}$. Obviously $|T| \geqslant \lambda$ since $\lambda_{n+1} < \lambda$. From Lemma 2.5 we obtain a cardinal λ' with $\lambda_{n+1} < \lambda' < \lambda$ and $T' \subseteq T$ such that $T' \subseteq \hat{B}_\lambda$, and $|T'| \geqslant (\lambda_{n+1}^{\aleph_0})^+$. So we can choose any $\lambda_{n+2} \in \mathring{\mu}$ such that $\max(\lambda', \mu_{n+2}) < \lambda_{n+2}$ and any $A_{n+1} \subseteq T'$ such that $|A_{n+1}| = (\lambda_{n+1}^{\aleph_0})^+$. ∎

Finally we want to prove (3.1): By construction of G we have $|G| = \mu$. Let $U \subseteq G$ such that $|U| = \mu$ and G/U is cotorsion-free. Denote by φ the canonical epimorphism from G onto G/U. Choose any $b \in G$. We want to show that $b \in U$. There is a $\lambda \in \mathring{\mu}$ such that $\|b\| < \lambda$. From (3.7) we obtain a λ-big family $(A_n, n \in \omega)$ contained in U. Using the list $\mathscr{L}$ we also find $(\lambda, \alpha) \in \mathscr{L}$ such that $b = b_\alpha^\lambda$ and $A_n^\lambda = {}^n A_\alpha^\lambda$ for all $n \in \omega$. Therefore $(\lambda, \alpha) \in C$. In particular the set of elements ${}^\pi a_\alpha^\lambda = a_\pi + b\pi \in G$ for $\pi \in I_p$ satisfies condition (††). Since $\pi \in I_p$, let π be a p-adic limit of π^m where $\pi^m \in \mathbb{Z}$. We will assume $p^m \mid (\pi^m - \pi)$. Since $a_\pi \in \prod_{n \in \omega} A_n p^n$, choose a finite sum $a_\pi^n = \sum_{i=1}^{n-1} a_\pi|_i$. Clearly for all n we have $p^n | (a_\pi - a_\pi^n)$ and also $p^m | (\pi b - \pi^m b + a_\pi - a_\pi^m)$ in $\hat{B}$. Let $q_m \in \hat{B}$ be the quotient which is unique since $\hat{B}$ is torsion-free. Therefore $p^m \cdot q_m = \pi b - \pi^m b + a_\pi - a_\pi^m \in G$ since $\pi^m b \in G$, $\pi b + a_\pi \in G$ and $a_\pi^m \in \prod_{i=1}^m A_i \subseteq U \subseteq G$. Because G is a pure subgroup of $\hat{B}$, we conclude also $q_m \in G$. Therefore

(a) $\pi b - \pi^m b + a_\pi - a_\pi^m \in p^m G$.

By definition of φ we also have $\varphi(\pi b - \pi^m b + a_\pi - a_\pi^m) = \varphi(\pi b + a_\pi) - \varphi(\pi^m b + a_\pi^m) = \varphi(\pi b + a_\pi) - \pi^m \varphi(b) \in p^m \varphi(G) = p^m(G/U)$ using (a) and $a_\pi^m \in U$. Since G/U is cotorsion-free, in particular G/U is reduced and therefore

(b) $\varphi(\pi b + a_\pi) = \pi\varphi(b)$ from the last equation.

Because $I_p \varphi(b) = \{\pi\varphi(b), \ \pi \in I_p\} = \{\varphi(\pi b + a_\pi), \ \pi \in I_p\} \subseteq G/U$ and G/U is cotorsion-free we derive $I_p \varphi(b) = 0$. Here we used that I_p is cotorsion and the fact that the class of cotorsion groups is closed under taking epimorphic images. Now $1 \in I_p$ implies $\varphi(b) = 0$ which is equivalent to saying that $b \in U$. It remains to show that G is cotorsion-free and homogeneous. Since G is a pure subgroup of the homogeneous group $\hat{B}$, we already know that G is homogeneous and the type is determined by the topology on $\hat{B}$. Because G is a pure subgroup of $\hat{B}$ we only have to show that $I_p \not\subseteq G$. Then we conclude from Theorem 1, that G is cotorsion-free. Assume that $\sigma: I_p \to G$ is a homomorphism. If $1 \in I_p$ is the identity of I_p and $x \in I_p$ we have $\sigma(x) = \sigma(x \cdot 1) = x\sigma(1)$ from the continuity of homorphism in the p-adic topology. We may assume without loss of generality that $\sigma(1) \in F$, where $G = F_*$ as in (3.5). Therefore $\sigma(1) = \sum_{i=1}^n e_i r_i$ for some $r_i \in \mathbb{Z}$ and some free generators e_i mentioned in (3.5)(a). If $x \in I_p$ also $\sigma(x) = x\sigma(1) = \sum_{i=1}^n x \cdot e_i r_i \in G$ and hence $s_x \sigma(x) = \sum_{i=1}^n s_x x r_i e_i = \sum_{i=1}^m c_i e_i \in F$ for $s_x \in \mathbb{Z}$ and some $c_i \in \mathbb{Z}$. Since the elements e_i are free generators also over

I_p, we have $s_x x r_i e_i = c_i e_i$. Assume that $(\lambda, \alpha) = (\lambda_1, \alpha_1) \geqslant (\lambda_i, \alpha_i)$, $e_i = \pi_i b_i + {}^{\pi_i}a_{\alpha_i}^{\lambda_i}$ for $i = 1,..., n$ and $\pi_1 = \pi$. Therefore we can find $\rho < \lambda$ such that $s_x x r_1 e_1|_\beta = s_x x r_1 a^\beta = c_1 a^\beta$ for $a^\beta = {}^\pi a_\alpha^\lambda|_\beta$ and for all $\rho < \beta < \lambda$. Since B is free, we obtain $s_x x r_1 = c_1 \in \mathbb{Q}_p \cap s_x \cdot I_p = s_x \mathbb{Q}_p$. Therefore $x r_1 \in \mathbb{Q}_p$ for all $x \in I_p$ which implies $r_1 = 0$. Using induction we obtain $r_i = 0$ and therefore $\sigma(1) = 0$ and $\sigma = 0$. Therefore G is cotorsion-free.

4. Applications of Theorem 2 to Torsion Theories

Torsion theories have a natural setting in abelian categories, see [22, 31]. They provide a nice link to topologies on modules, e.g., to Gabriel topologies and to the Goldie topology, compare [31, Sects. 4, 5 and p. 148]. In order to answer some questions, we will restrict to torsion theories of abelian groups and use the following closure operators on classes $\mathscr{X}$ of abelian groups:

$S\mathscr{X}$ = all subgroups of $\mathscr{X}$-groups,
$Q\mathscr{X}$ = all epimorphic images of $\mathscr{X}$-groups,
$\oplus\mathscr{X}$ = all direct sums of $\mathscr{X}$-groups,
$\Pi\mathscr{X}$ = all cartesian products of $\mathscr{X}$-groups and
$E\mathscr{X}$ = all extensions of $\mathscr{X}$-groups by $\mathscr{X}$-groups.

Using another notation of Hall, $\{A, B\}\mathscr{X}$ denotes the smallest class which is closed under the operators A and B and which contains $\mathscr{X}$, see Robinson [24, Sect. 1]. Let $T = \{E, Q, \oplus\}$ and $F = \{E, \Pi, S\}$. We also say that $\mathscr{C} \perp \mathscr{F}$ if $\mathrm{Hom}(A, B) = 0$ for all $A \in \mathscr{C}$, $B \in \mathscr{F}$; see also Göbel and Prelle [14, p. 423]. Classes $\mathscr{X}$ with $T\mathscr{X} = \mathscr{X}$ are called *torsion classes* and classes $\mathscr{F}$ with $F\mathscr{F} = \mathscr{F}$ are called *torsion-free classes*. A pair $(\mathscr{C}, \mathscr{F})$ of classes $\mathscr{C}$ and $\mathscr{F}$ is called a torsion theory if the following holds

(i) $\mathscr{C} \cap \mathscr{F} = \varnothing$.

(ii) $Q\mathscr{C} = \mathscr{C}$ and $S\mathscr{F} = \mathscr{F}$.

(iii) If A is an abelian group, there is a short exact sequence $0 \to T \to A \to F \to 0$ with $T \in \mathscr{C}$ and $F \in \mathscr{F}$;

see Dickson [2, p. 224].

The basic results on torsion theories have been derived in [2 and 3]. In particular, $(\mathscr{C}, \mathscr{F})$ is a torsion theory if and only if $\mathscr{C} \perp \mathscr{F}$ and $(\mathscr{C}, \mathscr{F})$ are maximal with respect to $\perp$. Necessarily $T\mathscr{C} = \mathscr{C}$ and $F\mathscr{F} = \mathscr{F}$, i.e., $\mathscr{C}$ is a torsion class and $\mathscr{F}$ is a torsion-free class. If $(T\mathscr{H}, F\mathscr{D})$ is a torsion theory, we denote $T\mathscr{H} = {}^\perp\mathscr{D}$ and $F\mathscr{D} = \mathscr{H}^\perp$ and say that $\mathscr{D}$ cogenerates $T\mathscr{H}$ and $\mathscr{H}$ generates $F\mathscr{D}$, compare [31, p. 139]. Similarly, we say that $\mathscr{H}$ generates $T\mathscr{H}$ and $\mathscr{D}$ cogenerates $F\mathscr{D}$. In particular, $(\mathscr{C}, \mathscr{C}^\perp)$ is a torsion theory if and only if $T\mathscr{C} = \mathscr{C}$ and $({}^\perp\mathscr{F}, \mathscr{F})$ is a torsion theory if and only if $F\mathscr{F} = \mathscr{F}$, cf. [31, p. 139, Proposition 2.1 and p. 140, Proposition 2.2]. If a torsion theory is generated and cogenerated by a set of abelian groups, it is also generated and cogenerated by a single abelian group. In this case (TC, FD), is a torsion theory for abelian groups C, D. The classical torsion theory is $(T\mathbb{Q}/\mathbb{Z}, F\mathbb{Q})$. Also hereditary torsion theories are generated and cogenerated by groups. A torsion theory $(\mathscr{T}, \mathscr{F})$ is hereditary if $S\mathscr{T} = \mathscr{T}$; compare Lambek [22, p. 6].

Using some algebraic notation, a collection of torsion theories (classes) is said to have small cardinality, if it is a set (and not just a class). Hereditary torsion theories have small cardinality; see Jans [18]. Therefore the questions arose, which are answered in the following

THEOREM 4.1. (a) *The torsion classes generated by a single abelian group do not have small cardinality.*

(b) *The torsion classes cogenerated by a single abelian group do not have small cardinality.*

(c) *The torsion classes not cogenerated by a single abelian group do not have small cardinality.*

(d) *The class of cotorsion-free abelian groups is not cogenerated by a single abelian group.*

Remarks. (4.1) was proved to be consistent with ZFC in a recent paper by Dugas and Herden [7, Corollary 2.1]; in fact (4.1) was derived from $\mathrm{ZFC} + V = L$. Hence the problem came up to find the absolute result as stated in (4.1). Theorem 4.1(c) shows that a conjecture in Göbel and Wald [16, p. 221] is true. The theorem also has some topological consequences, which are discussed in [7].

Proof. In the following let $\mathscr{H}_\kappa$ be the class of all cotorsion-free abelian groups of cardinality less than or equal to κ and $\mathscr{H} = \bigcup_\kappa \mathscr{H}_\kappa$. Let $\mathscr{E}$ be the (well-ordered) class of all strong limit singular cardinals κ of cofinality $>\aleph_0$ and G_λ one of the cotorsion-free groups of cardinality $\lambda \in \mathscr{E}$ constructed in Theorem 3.1.

Proof of (a). Consider the collection $\mathscr{M}$ of all torsion classes, TG_κ for all $\kappa \in \mathscr{E}$. Let $TG_\kappa = TG_\lambda$ for $\kappa \leqslant \lambda \in \mathscr{E}$. By definition of T, there are non-trivial homomorphisms $\kappa^*: G_\kappa \to G_\lambda$ and $\lambda^*: G_\lambda \to G_\kappa$. If $\lambda > \kappa$, then $\mathrm{Hom}(G_\lambda, G_\kappa) = 0$ by Theorem 3.1. Therefore $\lambda = \kappa$ and $\mathscr{M}$ is a class.

Proof of (b). Consider $\mathscr{M} = \{{}^\perp G_\kappa, \kappa \in \mathscr{E}\}$. Then ${}^\perp G_\kappa \neq {}^\perp G_\lambda$ if $\lambda < \kappa \in \mathscr{E}$ from (3.1). Hence $\mathscr{M}$ is a class.

Proof of (c). Using Theorem 2 for two incomparable types, we obtain two semirigid classes $\mathscr{C}$ and $\mathscr{D}$ which are not sets such that $\mathscr{C} \perp \mathscr{D}$ and $\mathscr{D} \perp \mathscr{C}$. We assume w.l.o.g. that $X, Y \in \mathscr{C} \cup \mathscr{D}$, and $|X| < |Y|$ implies $Y \perp X$. If $Y \in \mathscr{D}$, let $\mathscr{C}(Y) = \mathscr{C} \cup Y$. Then we derive that

$$ {}^\perp\mathscr{C}(Y) \text{ is not cogenerated by a set of abelian groups.} \tag{*} $$

If (*) does not hold, there is a subset $\mathscr{N}$ of $\mathscr{C}(Y)$ such that ${}^{\perp}\mathscr{C}(Y) = {}^{\perp}\mathscr{N}$. In this case we derive

$$\operatorname{Hom}(C, U) \neq 0 \qquad \text{for all} \quad C \in \mathscr{C}(Y) \quad \text{and} \quad U \in \mathscr{N}. \tag{$\dagger$}$$

If $Y \notin \mathscr{N}$, then $\mathscr{N} \subseteq \mathscr{C}$ and $Y \in \mathscr{D}$. Therefore $\mathscr{D} \perp \mathscr{C}$ implies $Y \perp \mathscr{N}$ which contradicts (†), hence $Y \in \mathscr{N}$. Let $C \in \mathscr{C}$ such that $|C| > |Y|$. The group C exists because $\mathscr{C}$ is not a set. Therefore $\operatorname{Hom}(C, Y) = 0$, which also contradicts (†) because $C \in \mathscr{C}(Y)$ and $Y \in \mathscr{N}$. Therefore (*) is shown.

Because $\mathscr{D}$ is also a class and not a set, it remains to show that

$${}^{\perp}\mathscr{C}(Y) \neq {}^{\perp}\mathscr{C}(W) \qquad \text{for all} \quad Y, W \in \mathscr{D} \quad \text{with} \quad |Y| \neq |W|. \tag{**}$$

We assume that $|Y| < |W|$. Then $W \perp Y$ and $W \perp \mathscr{C}$ imply $W \in {}^{\perp}\mathscr{C}(Y)$ and trivially $W \notin {}^{\perp}\mathscr{C}(W)$. ▮

Proof of (d). If $\mathscr{H}$ is cogenerated by a single group H, then $\mathscr{H} = FH$. In particular $H \in \mathscr{H}$ and therefore H is cotorsion-free. Choose $\kappa \in \mathscr{E}$ such that $\kappa > |H|$. Therefore $\operatorname{Hom}(G, H) = 0$ from (3.1). Since $X \in FH$ implies $\operatorname{Hom}(X, H) \neq 0$, we conclude $G_\kappa \notin FH$. On the other hand $G_\kappa \in \mathscr{H}$, since G_κ is cotorsion-free. Therefore $\mathscr{H} \neq FH$ and (d) is shown.

The existence of torsion classes also implies the existence of torsion classes generated by set, as follows from

THEOREM 4.2. *For a category of R-modules the following are equivalent*:

(1) *For every cardinal λ there is a family of λ distinct torsion classes.*

(2) *For every cardinal λ there are R-modules G_i for $i < \lambda$ such that $TG_i \neq TG_j$ if and only if $i \neq j$.*

(3) *There is a class of R-modules G_i for $i \in \mathbb{Q}$ such that $TG_i \neq TG_j$ if and only if $i \neq j$.*

Proof. Since $(3) \to (2) \to (1)$ is trivial, it remains to show $(1) \to (3)$. We first construct λ modules as in (3). By (1) there are distinct torsion classes A_t for $t < \lambda$.

If t, $s < \lambda$, choose

$$G_{ts} = \begin{cases} 0 & \text{if } A_t \subseteq A_s \\ G \in A_t \backslash A_s & \text{otherwise.} \end{cases}$$

Let $G_t^\lambda = \bigoplus_{s<\lambda} G_{ts}$ and observe that $TG_t^\lambda \subseteq A_t$ because $TA_t = A_t$ and $G_t \in A_t$. Since $0 \neq G_{ts} \in A_t \backslash A_s$, we conclude that $TG_t^\lambda \nsubseteq TG_s^\lambda$ if $A_t \nsubseteq A_s$. So there are λ distinct torsion classes TG_t^λ for $t < \lambda$. This construction can be made for an unbounded sequence Σ of cardinals λ. Using induction, there is a subset R_λ groups in $\{G_t^\lambda \mid t < \lambda\}$ such that the torsion classes TX, $X \in T_\lambda$

do not appear in the construction of $TG_t^{\lambda'}$ for some $t < \lambda' < \lambda$, then $\bigcup_{\lambda \in \Sigma} R_\lambda$ is the required class of R-modules in (3).

The following is an immediate corollary from Theorem 2:

COROLLARY 4.3. *For every cardinal λ there are groups G_i for $i < \lambda$ such that $G_i \notin T\{G_j, j < \lambda, j \neq i\}$. These groups can be chosen of cardinality μ for any strong limit singular cardinal $\mu \geqslant \lambda$ with cofinality of μ larger than ω.*

The assertion in (4.3) can be sharpened by (4.2)(3) and from (4.4) it follows, that this is even best possible.

Remark 4.4 (*ZFC + Vopenka principle*). Every torsion class is generated by a set.

Proof. If T is a torsion class which is not generated by a set or equivalently by a single group, then we define inductively groups

$$G_\alpha \in T \text{ for all ordinals } \alpha \text{ such that } G_\nu \perp G_\alpha \text{ for all } \nu < \alpha. \qquad (*)$$

If G_α is constructed for all $\alpha < \beta$, let $G = \bigoplus_{\alpha < \beta} G_\alpha$. Then $G \in T$ but G cannot generate T. Hence we find $H \in T$ such that $H \notin {}^\perp(G^\perp) = T'$. Let H_0 be the maximal T'-torsion subgroup of H. Then $0 \neq G_\beta = H/H_0 \in T \cap T'$, i.e., G_β is T'-torsion-free and $G \perp G_\beta$ implies $G_\alpha \perp G_\beta$ for all $\alpha < \beta$. Therefore $(*)$ is shown.

Selecting a subclass and changing names, we obtain from $(*)$ a class $G_\alpha \in T$ for all ordinals α such that $|G_\alpha| < |G_\beta|$ and $G_\alpha \perp G_\beta$ if $\alpha < \beta$. Hence there is no momomorphism $G_\alpha \to G_\beta$ if $\alpha \neq \beta$. This is in conflict with the Vopenka principle.

Finally we also remark, that Theorem 2 provides examples to a problem of Fuchs concerning balanced subgroups. If G is as in (3.1) and of type $\mathbb{Z}$, then the only balanced subgroups U of G have cardinality $|U| < |G|$.

5. RIGID CLASSES OF COTORSION-FREE ABELIAN GROUPS

If there is a huge cardinal κ, then Vopenka's principle (VP) is consistent; cf. Jech [19, p. 415]. By definition of (VP) there are no rigid classes in $ZFC + VP$; cf also T. Jech [19, p. 414]. This already illustrates that the existence of rigid classes corresponds to the non-existence of certain large cardinals. In a model of ZFC + $V = L$ there are no measurable cardinals by Scott's theorem; cf. Jech [19, p. 311]. The existence of a rigid class in $V = L$ is shown in Dugas and Herden [7] using methods from Dugas and Göbel [4] developed from Eklof and Mekler [9] and Shelah [28].

As indicated, we will derive the existence of rigid classes by excluding only large cardinals. To be precise, we consider the following principle

($\diamond \cap \mathrm{cf} = \omega$) There is a class of regular cardinals λ and subsets $S_\lambda \subseteq \{\delta < \lambda,\ \mathrm{cf}(\delta) = \omega\}$ such that S_λ is a stationary subset of λ which satisfies the diamond $\diamond_{S_\lambda}$, however $S_\lambda \cap \lambda'$ is not stationary in λ' for all limits $\lambda' < \lambda$.

Remark. Let $(\Phi \cap cf = \omega)$ be the principle as above, replacing the diamond $\diamond_{S_\lambda}$ by the weak diamond Φ_{S_λ}.

The set zero-sharp $0^{\#}$ is defined and the axiom "$0^{\#}$ exists" is discussed in Jech [19, p. 337–339]. Hence we will exclude large cardinals by means of exclusion of $0^{\#}$.

In Shelah [30] the following is shown.

THEOREM 5.1. *If V is a model of $\mathrm{ZFC} + \nexists 0^{\#}$, then V satisfies $(\diamond \cap \mathrm{cf} = \omega)$; in fact the class of regular cardinals λ in V to satisfy $(\diamond \cap \mathrm{cf} = \omega)$ can be chosen as the class of all successor cardinals of strong limit singular cardinals of cofinality $>\omega$.*

In view of (5.1) we will assume $(\diamond \cap \mathrm{cf} = \omega)$. The existence of a rigid system then follows by an argument similar to [4, 7, 28].

COROLLARY 5.2 $(\Phi \cap \mathrm{cf} = \omega)$. *If G is a cotorsion-free abelian group, there is a class $\mathscr{C}$ of strongly $|C|$-free groups C such that $\mathscr{C} \cup \{G\}$ is rigid.*

Proof. Use the proof given in [7] and observe that only $(\Phi \cap \mathrm{cf} = \omega)$ is needed. As in Section 3 we obtain a semi-rigid class by construction. However, this class is automatically also rigid, because the groups in $\mathscr{C}$ have only free subgroups of smaller cardinality.

Using (5.2) and (5.1), we obtain the same results as in [7, 8] under the weaker hypothesis $\mathrm{ZFC} + \nexists 0^{\#}$:

We only summarize the facts relevant for $\mathrm{ZFC} + \nexists 0^{\#}$:

(a) The torsion classes which are not generated by a single abelian group do not have small cardinality.

(b) Torsion classes which are generated and cogenerated by a single abelian group are well-determined and belong to the list of $2^{\aleph_0}$ members as in [8, Theorem 2.1].

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